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Transformations of path integrals

K S Narain† and Jan Tarski‡

† International Centre for Theoretical Physics, Trieste, Italy
‡ Scuola Internazionale Superiore di Studi Avanzati, Trieste, Italy and International Centre for Theoretical Physics, Trieste, Italy

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Abstract. Several examples of changes of variables in Feynman-type integrals are considered. In particular, path integrals in curvilinear coordinates (of a certain class) are rigorously defined. An appendix is devoted to path integrals for vector potentials.

1. Introduction

This article deals with changes of variables in path integrals, in particular with nonlinear transformations. We offer here a rigorous analysis which is based on the definition given in (Tarski 1979), and on subsequent extensions. However, we consider also curvilinear coordinates, leading to integrals for which this definition does not apply. Modified definitions are then needed.

The available rigorous results on nonlinear transformations in functional integrals are (apparently) still very limited. We note in particular the investigations of Gross (1960) and of Ramer (1974) dealing with positive-definite (measure-theoretic) Gaussian integrals.

Heuristic treatments of nonlinear transformations of path integrals, for contrast, are frequently encountered in gauge theories and elsewhere (cf e.g. Abers and Lee 1973). Such studies have also been made in the context of one-particle theory. We mention in particular the works of Fanelli (1975, 1976) on canonical transformations, and of Arthurs (1969), the latter being devoted to polar coordinates. (Investigations dealing with path integrals over curved spaces constitute a related body of material, cf. e.g. the recent article (Tarski 1982a), which contains further references.)

We give, in § 2, some elementary examples of integrals of quadratic functions, and of transformations of integrals. In § 3 we consider the problem of expressing path integrals in terms of curvilinear coordinates (but on a flat space rather than on a manifold). Appendix 1 contains the proofs of two propositions. Appendix 2 is devoted to vector potentials.

The content of the second appendix does not fall directly under the subject specified in the title. However, there are various points of contact between problems of vector potentials and those of curvilinear coordinates, and this circumstance induced us to include this appendix.

In particular, both situations give rise to questions about ordering of factors in path integrals. Such questions have been discussed many times, but heuristically. In the analysis that we present, ambiguities of ordering are largely eliminated.

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On the whole, the present article contains a collection of simple examples, which should clarify the general issues that we consider. We did not attempt to resolve any of the deeper problems related to transformations of path integrals.

It is convenient at this point to summarise the definition of Feynman-type integrals, following Tarski (1979). This definition yields an integral over an abstract real Hilbert space $\mathscr{H}$. The usual path integral is then obtained by specialising the Hilbert space to the space $\mathscr{H}_0$ of paths η parametrised by time, η: $[0, t] \to R^n$. This space is delimited by the scalar product $\langle \dot\eta, \dot\eta \rangle$ and the initial condition $\eta(0) = 0$. Note in this connection that $\frac{1}{2}m\langle \dot\eta, \dot\eta\rangle$ is just the kinetic part of the action.

If $\dim \mathscr{H} = k < \infty$, we set up the approximation

$$I^{b,\alpha}(f) = [(b - \mathrm{i}\kappa)/2\pi]^{k/2} \int \mathrm{d}^k u \exp(-\tfrac{1}{2}b\langle u - \alpha, u - \alpha\rangle) \exp(\tfrac{1}{2}\mathrm{i}\kappa\langle u, u\rangle) f(u), \tag{1.1}$$

where κ is a fixed parameter ($\operatorname{Im} \kappa \geq 0$, $\kappa \neq 0$; in applications κ will be the mass of the particle), $\alpha \in \mathscr{H}$, and $\operatorname{Re} b > 0$. We then take $\lim(b \to 0)$ of $I^{b,\alpha}(f)$, requiring that the limit must be the same along any non-tangential path, and must be independent of α. This latter condition ensures that in the limit, the integral is translationally invariant. With regard to the former condition, we will refer to it as the non-tangential limit.

If $\dim \mathscr{H} = \infty$, we consider sequences $\{P_j\}$ of increasing finite-dimensional projections P_j, satisfying $\lim_{j\to\infty} P_j = 1$. Then $f(P_j\xi)$, $\xi \in \mathscr{H}$ is a function on a finite-dimensional space, which we integrate as above (with α replaced by $P\alpha$). We denote the result by $I_{P_j}^{b,\alpha}(f)$. We then consider the limit of $I_{P_j}^{b,\alpha}(f)$ as $j \to \infty$ for suitable sequences of projections; i.e., we specify a family of sequences, and all sequences of this family must yield the same result. In Tarski (1979) we considered in particular the family $\hat{\mathscr{Q}}$ of all such sequences, and the subfamilies of form $\mathscr{Q}(P)$, defined by restricting the P_j by $P_j \geq P$, for all projections in all sequences.

The limit $j \to \infty$ is to be followed by $b \to 0$, with the same conditions as before. The result is the Feynman integral of f:

$$\int \mathscr{D}(\xi) \exp(\tfrac{1}{2}\mathrm{i}\kappa\langle \xi, \xi\rangle) f(\xi) = \lim_{b\to 0} \lim_{j\to\infty} I_{P_j}^{b,\alpha}(f). \tag{1.2}$$

The familiar polygonal approximation (e.g. Truman 1976) relates to this construction as follows. We first subdivide $[0, t]$, as e.g.

$$0 = t_0 < t_1 < \ldots < t_k = t. \tag{1.3}$$

The condition $\dot\eta = \text{constant}$ in each subinterval then determines a projection P, and one can show that $P \to 1$ as $\max(t_j - t_{j-1}) \searrow 0$ (Truman 1976, theorem 4). Note that for an approximation determined by such a projection, we may exploit

$$[\eta(t_j) - \eta(t_{j-1})]/(t_j - t_{j-1}) = \dot\eta(\tau) \qquad \text{if } t_{j-1} < \tau < t_j \tag{1.4}$$

to change the variables of integration from the $\dot\eta$'s to the $\eta(t_j)$. We will normally use the notation $\eta_j = \eta(t_j)$. (We recall also that $\eta_0 = 0$.) We state for reference the form of the approximation for the case $\alpha = 0$, denoting $F(P\eta)$ by $F_k(\eta_1, \ldots, \eta_k)$:

$$I_P^{b,0}(F) = \int \left(\prod_{j=1}^{k} \mathrm{d}^n\eta_j [(b - \mathrm{i}\kappa)/2\pi(t_j - t_{j-1})]^{kn/2} \right)$$
$$\times \exp[-\tfrac{1}{2}(b - \mathrm{i}m) \sum_j (\eta_j - \eta_{j-1})^2/(t_j - t_{j-1})] F_k(\eta_1, \ldots, \eta_k). \tag{1.5}$$

The function F_k may contain a factor $\delta(\eta_k - y)$, if one wishes to impose a second endpoint condition on the paths.

A sequence of subdivisions as in (1.3) defines a sequence of projections. We denote the family of sequences of this kind by $\mathcal{Q}_0$. This family is rather special as compared with the others that we mentioned, and lacks some of their properties. E.g. convergence with reference to $\mathcal{Q}_0$ does not guarantee rotational invariance of the integral. Consequently, convergence with reference to $\mathcal{Q}_0$ was not considered in Tarski (1979). We will make use of this family when constructing some modified forms of path integrals.

We will also encounter in the text path integrals over phase space. Here one integrates in effect over $\mathscr{H}_0 \dot{+} \mathscr{H}_0$, where the two spaces are related by the natural isomorphism $\dot{\eta}(\tau) \leftrightarrow p(\tau)$. These integrals (assumed over η or $\dot{\eta}$, and p, for definiteness) are characterised by the weight factor

$$\exp[\mathrm{i}\langle p, \dot{\eta}\rangle - \mathrm{i}(2m)^{-1}\langle p, p\rangle]. \tag{1.6}$$

Otherwise one proceeds as before, with projections to $P\mathscr{H}_0 \dot{+} P\mathscr{H}_0$ (following the options mentioned above), and with convergence factors $\exp(-\frac{1}{2}b\langle\dot{\eta} - \alpha, \dot{\eta} - \alpha\rangle - \frac{1}{2}b_0\langle p - \alpha_0, p - \alpha_0\rangle)$, (see Tarski 1982b for further details).

2. Quadratic integrands and examples of transformations

One familiar formula for Gaussian functional integrals (positive-definite or Feynman type) is the following:

$$\int \mathscr{D}(\xi) \exp(\tfrac{1}{2}\mathrm{i}\kappa\langle\xi, \xi\rangle)\langle\xi, T\xi\rangle = (\mathrm{i}\kappa)^{-1}\mathrm{Tr}\,T. \tag{2.1}$$

Here T is a linear operator, and the integral is unambiguously defined if and only if T is of trace class. The above formula also has a counterpart for Gaussian integrands:

$$\int \mathscr{D}(\xi) \exp(\tfrac{1}{2}\mathrm{i}\kappa\langle\xi, \xi\rangle) \exp(\tfrac{1}{2}\mathrm{i}\kappa\langle\xi, T\xi\rangle) = [\det(1+T)]^{-1/2}. \tag{2.2}$$

We assume here for convenience that $(1+T) > 0$. Again, the determinant and the integral are both unambiguously defined if and only if T is of trace class (cf Shale 1962).

Trace-class operators and determinants are relevant when one considers a general transformation $K\colon \mathscr{H} \to \mathscr{H}$ in functional integrals. We assume that K is bijective, but not necessarily linear. The map $\xi \to D_\xi(K\zeta)$, where D_ξ is the Gâteaux derivative, defines a linear operator $\nabla K\zeta$. Then $\det(\nabla K\zeta)$, if it exists, is the Jacobian for the transformation K.

For reference we recall the following familiar facts. First, if a linear operator has the form of an integral operator L with a symmetric kernel $L(x, y)$,

$$(Lf)(x) = \int_S \mathrm{d}^n y L(x, y) f(y), \tag{2.3}$$

then we can call $\int \mathrm{d}^n y L(y, y)$ its 'formal trace'. If the operator is of trace class, then this formal trace agrees with the trace.

Second, if K is nonlinear, and we are given e.g. an expression for $(K\zeta)(x)$, then the Volterra derivative $\delta(K\zeta)(x)/\delta\zeta(y)$ is the kernel of $\nabla K\zeta$ at ζ:

$$D_\xi(K\zeta):\xi(x)\to\int \mathrm{d}^n y[\delta(K\zeta)(x)/\delta\zeta(y)]\xi(y). \tag{2.4}$$

Third, in certain cases the restriction to trace-class operators can be slightly relaxed and Hilbert–Schmidt operators B can also be allowed. For example, a Gaussian measure transformed by the operator $1+B$ (assumed non-singular) is equivalent to the original (Shale 1962), cf propostion 1 below. However, Hilbert–Schmidt operators cannot be admitted e.g. in (2.2).

We return to functional integrals. The analogy of $\mathscr{D}(\xi)$ to the Lebesgue measure suggests the following form for transformation of the integral:

$$\int \mathscr{D}(\xi)\exp(\tfrac{1}{2}\mathrm{i}\kappa\langle\xi,\xi\rangle)F(\xi)$$
$$=\int \mathscr{D}(\zeta)\exp(\tfrac{1}{2}\mathrm{i}\kappa\langle\zeta,\zeta\rangle)|\det(\nabla K\zeta)|\exp[\tfrac{1}{2}\mathrm{i}\kappa(\langle K\zeta,K\zeta\rangle-\langle\zeta,\zeta\rangle)]F(K\zeta). \tag{2.5}$$

We included the factor $\exp(\frac{1}{2}\mathrm{i}\kappa\langle\zeta,\zeta\rangle)$ and then its inverse, in order to put the last integral into the standard form. We presuppose for these integrals the definition (1.2).

We emphasise that the foregoing is a generic relation, one that remains to be investigated for different transformations K and functions F. It holds in particular for integrable functions F, if K is a translation within $\mathscr{H}$. It also holds if K is an orthogonal transformation of $\mathscr{H}$ (i.e. a rotation), provided we set $|\det\nabla K\zeta|=|\det K|=1$.

One could say that equation (2.5) provides a particularly convenient description of change of variables. For this reason we give two special cases in the following propositions. In the next section we will encounter a different kind of situation.

If K is linear, then $\nabla K\zeta=K$. Let us write for now $K=1+T$, and let T' be the transpose of T. Then (2.5) reduces to

$$\int \mathscr{D}(\xi)\exp(\tfrac{1}{2}\mathrm{i}\kappa\langle\xi,\xi\rangle)F(\xi)$$
$$=|\det(1+T)|\int \mathscr{D}(\zeta)\exp(\tfrac{1}{2}\mathrm{i}\kappa\langle\zeta,\zeta\rangle)\exp[\tfrac{1}{2}\mathrm{i}\kappa\langle\zeta,(T+T'+T'T)\zeta\rangle]F((1+T)\zeta). \tag{2.6}$$

Proposition 1. Let T: $\mathscr{H}\to\mathscr{H}$ be linear and of trace class, and such that $1+T$ is invertible (i.e. -1 is not an eigenvalue of T). Let μ be a measure on $\mathscr{H}$ such that, for some integer $n\geqslant 0$,

$$\int d|\mu|(\chi)(1+\|\chi\|)^n<\infty \tag{2.7a}$$

(where $|\mu|$ is the absolute variation of μ), and let F be given by

$$F(\xi)=\langle\zeta_1,\xi\rangle\ldots\langle\zeta_n,\xi\rangle\int \mathrm{d}\mu(\chi)\exp(\mathrm{i}\langle\chi,\xi\rangle) \tag{2.7b}$$

where $\zeta_j\in\mathscr{H}$, $\forall_j\xi$. Then the integral on the RHS of (2.6) exists, and this equation is valid.

A proof based on routine manipulations is given in appendix 1. We remark that integrands such as F are further discussed in Berg and Tarski (1981).

We also give an example of a nonlinear transformation, for the case dim $\mathscr{H}=1$. Let F be a bounded and (Feynman) integrable function. For example, F could be the Fourier transform of a bounded measure:

$$F(u)=\int \mathrm{d}\nu(v)\exp(\mathrm{i}uv), \qquad \int \mathrm{d}|\nu|(v)<\infty. \tag{2.8}$$

Let $f\in\mathscr{S}$ be such that $|f'|\leq$ constant <1, and consider

$$Kv=v+f(v). \tag{2.9}$$

Proposition 2. Equation (2.5) is valid for the case dim $\mathscr{H}=1$, K as in (2.9), and F bounded and integrable. In this case (2.5) reduces to

$$I(F)=\int \mathscr{D}(v)\exp(\tfrac{1}{2}\mathrm{i}\kappa v^2)\{\exp\tfrac{1}{2}\mathrm{i}\kappa[f^2(v)+2vf(v)]\}[1+f'(v)]F(v+f(v)). \tag{2.10}$$

Note that $|f'|<1$ implies that K is bijective.

Proof. Let $\alpha\in R^1$, let Re $b>0$, and consider the approximation

$$I^{b,\alpha}(F)-(-\mathrm{i}\kappa/2\pi)^{1/2}\int_{-\infty}^{\infty}\mathrm{d}v\exp[-\tfrac{1}{2}b(v-\alpha)^2]\exp[\tfrac{1}{2}\mathrm{i}\kappa v^2]F(v). \tag{2.11}$$

We make the change of variable $v\to Kv$. We then obtain an integral which is the desired approximation to the RHS of (2.10), except that the b-dependent convergence factor now is

$$\exp[-\tfrac{1}{2}b(v-\alpha)^2]\exp\{-\tfrac{1}{2}b[f^2(v)+2vf(v)-2\alpha f(v)]\} \tag{2.12}$$

instead of $\exp[-\tfrac{1}{2}b(v-\alpha)^2]$.

Let us replace b by b_1 in the second exponent, and let us denote the resulting approximating integral by $J(b,b_1)$. Then $J(b,b)$ has the same limit as $I^{b,\alpha}(F)$, and in order to justify (2.10) we need to show that $J(b,0)$ also yields $I(F)$ in the limit; i.e., we need to show that

$$\lim_{b\to 0}[J(b,b)-J(b,0)]=0. \tag{2.13}$$

The difference $J(b,b)-J(b,0)$ can be estimated as follows. The second exponent can be bounded by $1+|b|C_1$ for some constant C_1. The integral for $J(b,0)$ with the integrand replaced by its absolute value is bounded by $|b|^{-1/2}C_2$, for some constant C_2. Thus

$$|J(b,b)-J(b,0)|\leq(1+|b|C_1-1)|b|^{-1/2}C_2, \tag{2.14}$$

and (2.13) follows.

The following comment could be made here. By showing that the two convergence factors lead to the same result, we are able to refer to the original definition of the integral. We are then able to state a proposition about integrability in a neat way. But otherwise, we cannot give any compelling reason for preferring the convergence factor $\exp[-\tfrac{1}{2}b(v-\alpha)^2]$ to that in (2.12).

3. Path integrals in curvilinear coordinates

We consider a Schrödinger particle on R^n, with Cartesian coordinates η^j. We introduce curvilinear coordinates Q^j in the following way:

$$\eta^j = \varphi^j(Q) = Q^j + \varphi_0^j(Q), \qquad Q^j = (\varphi^{-1})^j(\eta), \qquad \varphi(0) = 0. \tag{3.1}$$

We assume that the correspondence $\eta \leftrightarrow Q$ is bijective and continuous. Other conditions will be specified later. Note that this correspondence resembles that of proposition 2. However, now the path components $\dot{\eta}^j$ are the main variables, and the effect of nonlinearity on the path integral is rather different.

We should like to transform the path integral, so that it would express integration over paths $Q(\tau)$. Since $\dot{\eta}^j = \dot{Q}^l \partial_l \varphi^j(Q)$, the integral should have the form

$$\int_{Q(0)=0} \mathscr{D}_g(Q) \exp[\tfrac{1}{2}\mathrm{i}mS_0'(Q)]F(Q)\delta(Q(t) - \check{Q}), \tag{3.2}$$

where we set

$$S_0'(Q) = \int_0^t d\tau\, \dot{Q}^j \dot{Q}^k g_{jk}(Q) \qquad \text{and} \qquad g_{jk}(Q) = \sum_l (\partial_j \varphi^l)(\partial_k \varphi^l). \tag{3.3}$$

In the foregoing $\mathscr{D}_g(Q)$ should include a contribution from the metric tensor, cf below, and for definiteness we included a second endpoint condition.

In case of a particle on R^1, $S_0'(Q)$ becomes

$$S_0'(Q) = \int_0^t \mathrm{d}\tau\, \dot{Q}^2 \varphi'(Q)^2, \qquad \text{i.e. } g_{11} = \varphi'(Q)^2. \tag{3.4}$$

The coordinate Q might then be described as non-uniform rather than curvilinear, η being a uniform coordinate. A geodesic would in each case be an interval. However, if a geodesic is naturally parametrised by the arc length σ, then $\eta(\sigma)$ is a linear function but $Q(\sigma)$ is not.

A basic question now is whether the definition given in the introduction could apply. The answer appears to be negative, for the following reason. Let us consider a particle on R^1. Then, following § 2 and especially equation (2.5), the gradient of $\dot{Q}\varphi'(Q)$ should be the identity plus a trace-class operator. But we have, in terms of $\varphi_0 = \varphi - Q$,

$$\begin{aligned} &\delta[\dot{Q}(\tau)\varphi'(Q(\tau))]/\delta\dot{Q}(\sigma) \\ &\quad = \delta(\tau-\sigma) + \delta(\tau-\sigma)\varphi_0'(Q(\tau)) + \dot{Q}(\tau)\varphi_0''(Q(\tau))[\delta Q(\tau)/\delta\dot{Q}(\sigma)]. \end{aligned} \tag{3.5}$$

This kernel defines an operator $1 + T_1 + T_2$, corresponding to the three summands. The second has an infinite trace, coming from $\delta(0)$, and this singularity cannot be cancelled by T_2, since the last derivative is 1 if $\tau < \sigma$ and 0 if $\tau > \sigma$. (One can similarly verify that $T_1 + T_2$ is not Hilbert–Schmidt, cf § 2).

We proceed therefore to give an independent definition, or construction, of the integral in (3.2). We note that the transformation (3.1) defines in a natural way a transformation of the Hilbert space $\mathscr{H}_0$ (which is in general nonlinear). The notions of projections, translations, etc. therefore remain meaningful. However, in view of the form of S_0', and of the need to modify $\mathscr{D}$, we expect neither rotational nor translational invariance in the original form. For this reason we ignore the more general scheme described in the introduction. Rather, we restrict our approximations to those which are defined by subdivisions of $[0, t]$, as in (1.3). For the convergence factors we take $\exp(-\frac{1}{2}bS_0')$, Re $b>0$, with the shift vectors α omitted.

Next, one knows that a polygonal approximation in the Q variables is in general inadequate for the kinetic part of the action, $\frac{1}{2}mS_0'$ (Edwards and Gulyaev 1964). We therefore resort to the following hybrid procedure. We approximate S_0' by employing geodescis parametrised by arc length. A contribution from one time subinterval then becomes

$$S_0'(t_j-t_{j-1};\,Q_j,\,Q_{j-1})=\|\varphi^{-1}(Q_j)-\varphi^{-1}(Q_{j-1})\|^2(t_j-t_{j-1})^{-1}, \tag{3.6}$$

i.e. equal to the polygonal approximation in the η-variables.

For the proper integrand F, or for $F\delta$, we use the polygonal approximation in the Q's. We will denote such an approximation of F by $F(PQ)$, as before, where the projection P corresponds to the chosen partition of $[0, t]$ (cf § 1). Now, we construct the following approximation, as modifications of (1.5):

$$I_\varphi(P, b; F)=\int\left\{\prod_{j=1}^{k} d^nQ_j g(Q_j)^{1/2}[(b-\mathrm{i}m)/2\pi(t_j-t_{j-1})]^{kr/2}\right\}$$

$$\times\exp[-\tfrac{1}{2}(b-\mathrm{i}m)\sum_j S_0'(t_j-t_{j-1};\,Q_j,\,Q_{j-1})]F(PQ)\delta(Q_k-\check{Q}). \tag{3.7}$$

Here $Q_0=0$, $t_0=0$, $t_k=t$, and $g(Q)=\det g_{jk}$, $g^{1/2}$ being the Jacobian of the transformation $\eta\to Q$. (Such factors occur also in path integrals over manifolds, cf Tarski (1982a).) We now define the integral in (3.2) by taking limits: $\max(t_j-t_{j-1})\searrow 0$, followed by $b\to 0$ non-tangentially, as in § 1.

Let us return to the approximation (3.7). If we express it in terms of the η-variables, we obtain (1.5), where however $F(P\eta)$ will be replaced by $F(\varphi P\varphi^{-1}\eta)$. We see that all we have done in effect is to introduce a 'distorted polygonal approximation'.

We emphasise that such approximations provide the only way we know at present of incorporating transformations φ, and the variables Q, into rigorous path integrals. It is this circumstance, rather than specific applications, that motivated our search for some properties of these approximations. Hence the following proposition, which we precede by a technical lemma.

Lemma 3. Let V be bounded and continuous potential (on R^n), and let φ be bijective, of class $\mathscr{C}^1$, and such that $g(Q)\neq 0$ everywhere. Let

$$\bar{f}(Q)=\exp\left(-\mathrm{i}\int_0^t d\tau\, V(\varphi(Q(\tau)))\right). \tag{3.8}$$

Then the values $|I_\varphi(P, b; \bar{f})|$ (cf (3.7)) are bounded by a quantity which is independent of P (but may depend on b, etc).

This lemma is proved in appendix 1.

Proposition 4. Under the hypotheses of lemma 3,

$$g^{-1/2}(\check{Q})\int_{Q(0)=0}\mathscr{D}_g(Q)\exp[\tfrac{1}{2}\mathrm{i}mS_0'(Q)]\bar{f}(Q)\delta(Q(t)-\check{Q})=G(t;\,\varphi^{-1}(\check{Q}), 0), \tag{3.9}$$

G being the Schrödinger Green's function (for the Cartesian coordinates η).

Proof. We express the approximations (3.7) in terms of η-variables, with $\bar{f}(\varphi PQ) = f(\varphi P\varphi^{-1}\eta)$. For $b-im>0$, the integral reduces to the Wiener integral. The bounded convergence theorem, together with the continuity of φ, f, and V, imply that we obtain the Wiener integral of $f(\eta)$ as $\max(t_j - t_{j-1}) \searrow 0$, i.e. $P \to 1$. Lemma 3 and Vitali's theorem then imply that the corresponding convergence extends to the half-plane $\mathrm{Re}(b-im)>0$. (For this method of proof cf Nelson (1966) and Tarski (1979).) We obtain in this way the (usual) analytically continued Green's function, and the limit as $b \to 0$ is justified as in Nelson (1966) and Tarski (1979).

We conclude with three comments. Firstly we observe that the foregoing approximations leave no free choice (or, no ambiguity) with regard to the ordering of factors in S_0'.

Secondly the integral (3.2) is often considered as a density in its dependence on the initial point Q_0 (here $Q_0 = 0$) and on $\check{Q}$. For this purpose, appropriate powers of $g(Q_0)$ and $g(\check{Q})$ could be adjoined. Our construction is in this respect somewhat asymmetrical, as equation (3.9) shows.

Thirdly the approach just sketched can be easily adapted to phase-space integrals. We extend the map $\eta = \varphi(Q)$ to $(p, \eta) = \Phi(\pi, Q)$, where π is the momentum canonical to Q. For brevity in writing we now assume a particle on R^1. Then $p = \pi\varphi'(Q)^{-1}$, so that:

$$(p, \eta) = \Phi(\pi, Q) = (\pi\varphi'(Q)^{-1}, \varphi(Q)). \tag{3.10}$$

Now given $f(p, \eta)$, we may set up the approximations by integrating $f(\Phi P\Phi^{-1}(p, \eta))$ over $P\mathscr{H}_0 \dotplus P\mathscr{H}_0$ with convergence factors, cf introduction and Tarski (1982b). Here $P(\pi, Q)$ has the same meaning as $P(p, \eta)$, i.e. is defined by averaging π as well as $\dot{Q}$ in a subinterval. If the limits $\max(t_j - t_{j-1}) \searrow 0$ and then b, $b_0 \to 0$ exist, with suitable conditions, then these limits could define the phase-space path integral of f.

This procedure differs from the 'midpoint convention', employed in Arthurs (1969) and elsewhere. It would be of interest to relate the two.

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Appendix 1. Two proofs

Proof of Proposition 1. First we show that $F((1+T)\xi)$ is the Fourier transform of a transformed measure μ', under the hypothesis that $F(\xi)$ is the Fourier transform of μ. We use the criterion of Gross (1963). Clearly $F((1+T)\xi)$ is positive definite. Next, $F(\xi)$ is necessarily of the form $F_0(B\xi)$ with F_0 continuous and B Hilbert–Schmidt. But then $F((1+T)\xi) = F_0(B(1+T)\xi)$, and $B(1+T)$ is likewise Hilbert–Schmidt (Gelfand and Vilenkin 1964). It follows that $F((1+T)\xi)$ is the Fourier transform of a bounded measure μ'. This measure clearly satisfies, for any integrable f,

$$\int \mathrm{d}\mu'(\chi)f(\chi) = \int \mathrm{d}\mu(\chi)f((1+T')\chi). \tag{A1.1}$$

Next, the linear factors on the RHS become $\langle \zeta_j', \xi\rangle$ where $\zeta_j' = (1+T')\zeta_j$. We recall, moreover, that such factors result from applying $i^{-1}D_{\zeta_j'}$ inside the μ'-integral in (A1.1), if $f(\chi) = \exp(\mathrm{i}\langle\chi, \xi\rangle)$. (Here D_ζ is the Gâteaux derivative with respect to χ.) The integral on the RHS of (2.6), i.e. without the determinant, now becomes

$$\mathrm{i}^{-n}\int \mathscr{D}(\xi)\exp[\tfrac{1}{2}\mathrm{i}\kappa\langle\xi, (1+T')(1+T)\xi\rangle]\int \mathrm{d}\mu'(\chi)D_{\zeta_1'}\dots D_{\zeta_n'}\exp(\mathrm{i}\langle\chi, \xi\rangle). \tag{A1.2}$$

We consider now an approximation, determined by a projection P and a convergence factor (the indicated interchange of operations being clearly allowed):

$$\begin{aligned}
\mathrm{i}^{-n}\int \mathrm{d}\mu'(\chi)D_{\zeta_1'}\dots D_{\zeta_n'}&[(b-\mathrm{i}\kappa)/2\pi]^{k/2}\int \mathrm{d}^k u\,\exp(-\tfrac{1}{2}b\langle u-P\alpha, u-P\alpha\rangle)\\
&\times\exp(\tfrac{1}{2}\mathrm{i}\kappa\langle u, u\rangle)\exp\langle u, [\mathrm{i}\kappa(T+T'+T'T)\restriction P\mathscr{H}]u\rangle\exp(\mathrm{i}\langle P\chi, u\rangle)\\
&= \mathrm{i}^{-n}\exp(-\tfrac{1}{2}b\langle P\alpha, P\alpha\rangle)\{\det[1-\mathrm{i}\kappa(b-\mathrm{i}\kappa)^{-1}P(T+T'+T'T)P]\}^{-1/2}\\
&\quad\times\int \mathrm{d}\mu'(\chi)D_{\zeta_1'}\dots D_{\zeta_n'}\exp(\tfrac{1}{2}\langle bP\alpha+\mathrm{i}P\chi,\\
&\quad\times\{[b-\mathrm{i}\kappa-\mathrm{i}\kappa(T+T'+T'T)]\restriction P\mathscr{H}\}^{-1}(bP\alpha+\mathrm{i}P\chi)\rangle).
\end{aligned} \tag{A1.3}$$

(In the last expression the scalar product is bilinear and symmetric, but not Hermitian.) We note that this evaluation is as in Tarski (1979), proof of proposition 7, but with L replaced by $T+T'+T'T$, and $bP\alpha$ by $bP\alpha+\mathrm{i}P\chi$.

We already see from (A1.3) what the two limits $P\to 1$ (rather, $j\to\infty$ where P_j is a sequence which $\to 1$) and $b\to 0$ will yield. To justify these limits, we refer to the analogous arguments in the iterature, and will not reproduce the details here. In particular: for $P\to 1$, cf Tarski (1979). For $b\to 0$, cf Tarski (1979), proof of proposition 5, or Berg and Tarski (1981). We remark that the justification of this limit in the cited works, i.e. for $T=0$, depends on certain inequalities. These inequalities extend to the present case, since both $1+T$ and $(1+T)^{-1}$ are bounded. We observe that integrability on RHS of (2.6) now follows.

The two limits in question reduce (A1.2) to

$$\mathrm{i}^{-n}[\det(1+T')(1+T)]^{-1/2}\int \mathrm{d}\mu'(\chi)D_{\zeta_1'}\dots D_{\zeta_n'}\exp[\tfrac{1}{2}(\mathrm{i}\kappa)^{-1}\langle\chi, (1+T)^{-1}(1+T')^{-1}\chi\rangle]. \tag{A1.4}$$

The determinant factor equals $|\det(1+T)|^{-1}$. As to the action of the operators $D_{\zeta_1'}$ note that $D_{\zeta_n'}$ yields

$$(\mathrm{i}\kappa)^{-1}\langle(1+T')\zeta_n, (1+T)^{-1}(1+T')^{-1}\chi\rangle = (\mathrm{i}\kappa)^{-1}\langle\zeta_n, (1+T')^{-1}\chi\rangle. \tag{A1.5}$$

In accordance with (A1.1), we replace μ' by μ and χ by $(1+T')\chi$. Then in particular T disappears from (A1.5), and so the action of $D_{\zeta_n'}$ becomes equivalent to that of $D_{\zeta_j'}$ on $\exp[\tfrac{1}{2}(\mathrm{i}\kappa)^{-1}\langle\chi, \chi\rangle]$. The analogous conclusion applies to the other $D_{\zeta_j'}$. Thus (A1.4) becomes

$$|\det(1+T)|^{-1}\mathrm{i}^{-n}\int \mathrm{d}\mu(\chi)D_{\zeta_1}\ldots D_{\zeta_n}\exp[\tfrac{1}{2}(\mathrm{i}\kappa)^{-1}\langle\chi,\chi\rangle]. \tag{A1.6}$$

The factor $|\ldots|^{-1}$ compensates the determinant that was dropped in forming (A1.2), and the part $i^{-n}\ldots$ equals the LHS. Thus (2.6) is established.

Proof of Lemma 3. We reintroduce

$$f(\eta)=\exp\left[-\mathrm{i}\int_0^t \mathrm{d}\tau\, V(\eta(\tau))\right]=\bar{f}(\varphi^{-1}(\eta)). \tag{A1.7}$$

An approximation $I_\varphi(P,b;\bar{f})$ will then be determined by the function $f(\varphi P\varphi^{-1}\eta(\tau))$, cf the discussion preceding the statement of the lemma. If $t_{j-1}\leqslant\tau\leqslant t_j$, this function is given by

$$f(\varphi P\varphi^{-1}\eta(\tau))=f\left(\varphi\left[\frac{t_j-\tau}{t_j-t_{j-1}}\varphi^{-1}(\eta_{j-1})+\frac{\tau-t_{j-1}}{t_j-t_{j-1}}\varphi^{-1}(\eta_j)\right]\right). \tag{A1.8}$$

We consider now, with $y=\varphi(\check{Q})$,

$$\begin{aligned}I_\varphi(P,b;\bar{f})=g(y)^{1/2}\int&\left\{\prod_{j=1}^k d^n\eta_j[(b-\mathrm{i}m)/2\pi(t_j-t_{j-1})]^{nk/2}\right\}\\&\times\exp[-\tfrac{1}{2}(b-\mathrm{i}m)\sum_j\|\eta_j-\eta_{j-1}\|^2/(t_j-t_{j-1})]\\&\times\exp\left[-\mathrm{i}\int_0^t\mathrm{d}\tau\, V(\varphi P\varphi^{-1}\eta(\tau))\right]\delta(\eta_k-y).\end{aligned} \tag{A1.9}$$

Since V is bounded, say $|V|\leqslant C$, we may expand the exponential of $\int\mathrm{d}\tau V$ and interchange integrations. Consider the term

$$\begin{aligned}I_N:=\int\prod d^n\eta_j\ldots\left(-\mathrm{i}\int\mathrm{d}\tau\, V\right)^N\delta=(-\mathrm{i})^N\int\mathrm{d}\tau_1\ldots\mathrm{d}\tau_N\int\prod\mathrm{d}^n\eta_j\ldots\\ \times V(\ldots\tau_1)\ldots V(\ldots\tau_N)\delta(\eta_k-y).\end{aligned} \tag{A1.10}$$

In view of (A1.8), the N values of the $V(\ldots\tau)$, together with $\delta(\eta_k-y)$, depend on at most $2N+1$ of the η_j's. We utilise, therefore, a device which was employed in the proof of proposition 24 in Tarski (1979) (and is due originally to Friedrichs and Shapiro). Namely, we express I_N as a $n(2N+1)$-dimensional integral over suitable linear combinations of the η_j's, the remaining integrations reducing to unity. The underlying idea is, of course, that of splitting $P\mathscr{H}_0$ into two perpendicular subspaces, the V's and δ being non-trivial on one of these.

Suppose for the moment that $V=0$ ($f=1$). Then only the subspace with $\dot{\eta}=$-constant is relevant, and integrating over this subspace yields the analytically continued free-particle Green's function:

$$I_\varphi(P,b;1)=G^{(0)}_{b-\mathrm{i}m}(t;y,0)=[(b-\mathrm{i}m)/2\pi t]^{n/2}\exp[-(b-\mathrm{i}m)\|y\|^2/2t], \tag{A1.11a}$$

$$|I_\varphi(P,b;1)|=|G^{(0)}_{b-\mathrm{i}m}|=(|b-\mathrm{i}m|/\mathrm{Re}\, b)^{n/2}G^{(0)}_{\mathrm{Re}\, b}(t;y,0). \tag{A1.11b}$$

We can see from these relations that, whenever the absolute value of an l-dimensional integral (of the kind as in (A1.9)) is to be taken, then the following is to be done:

first, the coefficient $b-im$ in the exponent should be replaced by Re b. Then, since the normalising factor corresponding to this Re b requires $(\mathrm{Re}\, b)^{l/2}$ rather than $(b-\mathrm{i}m)^{l/2}$, one should replace $(b-\mathrm{i}m)^{l/2}$ by $(|b-\mathrm{i}m|/\mathrm{Re}\, b)^{l/2}$.

Now, in view of our previous remark, for I_N we need to estimate a $(2N+1)n$-dimensional integral, or a $2Nn$-dimensional integral in addition to that for (A1.11). Therefore

$$|I_N| \leq t^N C^N (|b-\mathrm{i}m|/\mathrm{Re}\, b)^{Nn} |G^{(0)}_{b-\mathrm{i}m}(t;\, y, 0)|, \tag{A1.12}$$

from which

$$\begin{aligned} |I_\varphi(P, b; \bar{f})| &\leq g(y)^{1/2} \sum |I_N|/N! \\ &\leq g(y)^{1/2} \exp[tC(|b-\mathrm{i}m|/\mathrm{Re}\, b)^n] |G^{(0)}_{b-\mathrm{i}m}(t;\, y, 0)|. \end{aligned} \tag{A1.13}$$

This is the desired bound.

Appeindix 2. Vector potentials

The path-integral representations of Schrödinger Green functions in the presence of external vector potentials A are familiar (e.g. Feynman and Hibbs 1965), and take the form

$$G(t;\, y, 0) = \int_{\eta(0)=0} \mathscr{D}(\eta) \exp[\tfrac{1}{2}\mathrm{i}m\langle\dot\eta, \dot\eta\rangle + \mathrm{i}e\langle\dot\eta, A(\eta)\rangle]\delta(\eta(t)-y) \tag{A2.1a}$$

$$\begin{aligned} &= \int_{\eta(0)=0} \mathscr{D}(\eta)\mathscr{D}(p) \exp[\mathrm{i}\langle p, \dot\eta\rangle - \mathrm{i}(2m)^{-1}\langle p-A(\eta), p-A(\eta)\rangle] \\ &\quad \times \delta(\eta(t)-y), \end{aligned} \tag{A2.1b}$$

where the scalar products have the obvious interpretation,

$$\langle\dot\eta, A(\eta)\rangle = \int_0^t \mathrm{d}\tau \sum_j \dot\eta^j(\tau) A_j(\eta(\tau)), \text{etc.} \tag{A2.2}$$

Note that we can reduce (A2.1b) to (A2.1a) by making the (heuristic) change of variables $p \to p + eA$ and then doing the ξ-integral.

The foregoing expressions were considered on a number of occasions, primarily as examples illustrating the problem of ordering of factors, or the connection with stochastic integrals (cf e.g. McLaughlin and Schulman 1971). In this section we consider the integrals in (A2.1) from the point of view of the definition (1.2) and its phase–space analogue. We will see that for a pure gauge field, the ordering problem is automatically resolved by the above formulation of path integrals (in contrast to the treatment of McLaughlin and Schulman (1971), but we will also give an example of an integral which does not converge in a basis-independent way.

Let us consider a potential $A = \nabla\lambda$, a pure gauge field. Let P project onto functions where $\dot\eta(\tau) = \text{constant}$, and let $P' \geq P$ (i.e. we take the family $\mathscr{Q}(P)$ of sequences of projections). Then in the approximation defined by P',

$$\int_0^t \mathrm{d}\tau(\mathrm{d}P'\eta/\mathrm{d}\tau)\nabla\lambda(P'\eta) = \int \mathrm{d}\tau(\mathrm{d}/\mathrm{d}\tau)\lambda(P'\eta(\tau)) = \lambda(P'\eta(t)) - \lambda(P'\eta(0)). \tag{A2.3}$$

However, $P'\eta(0) = 0$, and by using $P\eta(t) = \eta(t)$ we easily conclude $P'\eta(t) = \eta(t)$. The presence of $\delta(\eta(t)-y)$ in the integrand now yields $\lambda(y) - \lambda(0)$. After removing the factor $\exp(ie[\lambda(y)-\lambda(0)])$, we obtain the integral for the free-particle Green function, for any such P'.

We remark that the potential $A=\nabla\lambda$ can also be easily handled in case of the phase-space integral (A2.1*b*). Indeed, if P' now defines an approximation to (A2.1*b*), and $P'\geqslant P$, then we can make the change of variables $P'p\to P'p+A(P'\eta)$, integrate over $P'p$, and proceed as before.

It is instructive to verify (A2.3) in a different way for the case of a particle on R^1 with $A(\eta)=\eta$. Then

$$\langle\dot\eta,\eta\rangle=:\langle\dot\eta,B_1\dot\eta\rangle=\int_0^t \mathrm{d}\tau\,\mathrm{d}\tau'\;\dot\eta(\tau)\dot\eta(\tau')\dot\theta_\tau(\tau'),\tag{A2.4a}$$

i.e., B_1 is determined by the kernel $\dot\theta_\tau(\tau')$, which is given by

$$\begin{aligned}\dot\theta_\tau(\tau')&=1 &&\text{for}\quad \tau'<\tau,\\ &=0 &&\text{for}\quad \tau'>\tau\end{aligned}\tag{A2.4b}$$

(so that $\theta_\tau(\tau')=\min(\tau,\tau')$). Only the symmetric part of B_1 contributes to the scalar product, so that we may replace B_1 by $\frac12(B_1+B_1^t)$, with the kernel

$$b_1(\tau,\tau')=\tfrac12[\dot\theta_\tau(\tau')+\dot\theta_{\tau'}(\tau)]=\tfrac12\qquad(\text{AE})\tag{A2.5}$$

(AE: 'almost everywhere'). Therefore (in agreement with (A2.3)),

$$\langle\dot\eta,\eta\rangle=\tfrac12\int_0^t \mathrm{d}\tau\dot\eta(\tau)\int_0^t \mathrm{d}\tau'\dot\eta(\tau')=\tfrac12\eta^2(t)=\tfrac12 y^2.\tag{A2.6}$$

We next present the example of a particle on R^2, with $A(\eta)=L\eta$, L being a skew matrix:

$$\begin{aligned}\langle\dot\eta,A(\eta)\rangle&=:\langle\dot\eta,B_2\dot\eta\rangle\\ &=\int_0^t \mathrm{d}\tau(\dot\eta^x,\dot\eta^y)\begin{pmatrix}0&1\\-1&0\end{pmatrix}(\eta^x,\eta^y)^t\\ &=\int_0^t \mathrm{d}\tau\mathrm{d}\tau'(\dot\eta^x,\dot\eta^y)(\tau')\begin{pmatrix}0&\dot\theta_\tau(\tau')\\-\dot\theta_\tau(\tau')&0\end{pmatrix}\begin{pmatrix}\dot\eta^x\\ \dot\eta^y\end{pmatrix}(\tau').\end{aligned}\tag{A2.7}$$

We symmetrise the operator as in (A2.5), but now this requires a simultaneous interchanging $\tau\leftrightarrow\tau'$ and transposing the matrix. The kernel matrix becomes, in view of (A2.5),

$$b_2(\tau,\tau')=\tfrac12\begin{pmatrix}0&\dot\theta_\tau(\tau')-\dot\theta_{\tau'}(\tau)\\-\dot\theta_\tau(\tau')+\dot\theta_{\tau'}(\tau)&0\end{pmatrix}=\begin{pmatrix}0&\dot\theta_\tau(\tau')-\frac12\\-\dot\theta_\tau(\tau')+\frac12&0\end{pmatrix}.\tag{A2.8}$$

The eigenvectors (or some of them) can easily be found. Let n be an integer, and

$$v_n(\tau)=(\dot\eta^x,\dot\eta^y)^t(\tau)=[\sin(2n+1)\pi\tau/t,\cos(2n+1)\pi\tau/t]^t.\tag{A2.9a}$$

Then

$$\int_0^t \mathrm{d}\tau\,b_2(\tau,\tau')v_n(\tau')=(t/\pi)(2n+1)^{-1}v_n(\tau).\tag{A2.9b}$$

Since $\Sigma|2n+1|^{-1}$ diverges, and the trace class is determined by absolute convergence, we conclude that B_2 is not of trace class. Therefore (A2.1*a*) does not converge, and this conclusion extends easily to (A2.1*b*).

More generally, if $A(\eta) = L\eta$, where L is a constant matrix, then the symmetric part of L is a pure gauge field, while the skew part can be analysed by exploiting the above example. We summarise this below.

Proposition 5. If $A = \nabla\lambda$ (and A is continuous), then the path integrals in (A2.1a, b) exist, and are equal to the free-particle Green function times $\exp\{ie[\lambda(y) - \lambda(0)]\}$. (Convergence then is with reference to the family $\mathscr{Q}(P)$, where P projects onto constant $\dot{\eta}$.) If $A(\eta) = L\eta$ where L is a constant matrix having a non-zero skew part, then the integrals in (A2.1a, b) do not converge in a basis-independent way.

We conclude with the following comment. The case of a constant (non-zero) magnetic field corresponds to $A(\eta) = L\eta$ with L having a skew part. The path integrals could then be defined by specifying an additional limiting procedure, but we forego further discussion here. We may note, however, that in case of an analogous integrand in a Wiener integral, the theory of stochastic integrals would be applicable (*loc. cit.* and references given therein). An attempt to extend this theory to Feynman integrals is made in (Garczyński 1980).

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